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Characters and Artin L-functions

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§1. Introduction

We shall prove that for every profinite group its linear representation of finite dimension over the complex number field is lifted from a representation of its finite quotient group. (cf. Theorem 2.) This will imply that the multiplicative group of all the Artin L-functions is isomorphic to the additive group generated by characters of the absolute Galois group of the rational number field. (cf. Theorem 3.)

Gassmann, Komatsu, Perlis, and others showed that the number of algebraic number fields with the same Zeta function is always finite, and further that there exist non-conjugate fields with the same Zeta functions. Neukirch's conjecture, proved by Ikeda, Iwasawa, and Uchida in 1976 independently, says that algebraic number fields are conjugate if and only if their absolute Galois groups are isomorphic. Therefore the Zeta function does not determine the absolute Galois group.

In the latter half of this paper, we shall give similar results for Artin L-functions to Zeta functions. Namely there exist at most finitely many pairs of finite normal extensions and faithful characters of their Galois groups which correspond to the same Artin L-function. (cf. Theorem 4.) Moreover there exist normal extensions such that the sets of all their Artin L-functions are exactly the same while their Galois groups are non-isomorphic. (cf. Theorem 5.) In other wards only the Artin L-functions do not determine the Galois group of the extension.

§2. Artin L-functions

Throughout this paper $\mathrm{Gal}(E/K)$ means the Galois group of a normal extension E/K. Let E/K be a finite normal extension of algebraic number fields with $G=\mathrm{Gal}(E/K)$ and χ a character of a linear representation of G over the complex number field $\boldsymbol{C}$. Then the Artin L-function $L(s, \chi, E/K)$ can be defined for a pair $(\chi, E/K)$. Let $\mathrm{Irr}(G)$ be the set of all irreducible characters of G and $\psi=\sum a(\chi)\chi$ $(\chi\in\mathrm{Irr}(G))$ be a generalized character of G, where $a(\chi)\in\boldsymbol{Z}$. Then we can also define the *Artin L-function attached to the generalized character* ψ by

$$L(s, \psi, E/K)=\prod_{\chi} L(s, \chi, E/K)^{a(\chi)} ,$$

because Artin L-function is additive.

For a prime ideal $\mathfrak{p}$ of K, we take a prime ideal $\mathfrak{P}$ of E which divides $\mathfrak{p}$. Let $I_{\mathfrak{P}}$ denote the inertia group of $\mathfrak{P}$ and $\sigma_{\mathfrak{P}}$ a Frobenius substitution of $\mathfrak{P}$. The cardinal number of a set S shall usually be denoted by $|S|$. For a positive rational integer m, we put

$$\psi(\mathfrak{p}^m)=|I_{\mathfrak{P}}|^{-1} \sum \psi(\sigma_{\mathfrak{P}}^m \tau) \quad (\tau \in I_{\mathfrak{P}}) .$$

Then it follows that

$$\log L(s, \psi, E/K)=\sum_{\mathfrak{p}} \sum_{m=1}^{\infty} \psi(\mathfrak{p}^m)/mN(\mathfrak{p})^{ms}$$

for $\mathrm{Re}\,(s)>1$, where $\mathfrak{p}$ runs over all prime ideals of K and $N(\mathfrak{p})$ means the absolute norm of $\mathfrak{p}$. The following theorem is well-known (cf. Artin [1]) but we give a proof here since it plays an important role in this paper.

THEOREM 1. *If the base field K coincides with the rational number field $\boldsymbol{Q}$, then the Artin L-functions $L(s, \chi, E/\boldsymbol{Q})$ for irreducible characters χ of G are multiplicatively independent. Namely if a functional equation*

$$\prod L(s, \chi, E/\boldsymbol{Q})^{a(\chi)}=1 \quad (\chi \in \mathrm{Irr}\,(G))$$

holds with $a(\chi) \in \boldsymbol{Z}$, then we always have $a(\chi)=0$ for all $\chi \in \mathrm{Irr}\,(G)$.

Proof. Let $\psi=\sum a(\chi)\chi$ $(\chi \in \mathrm{Irr}\,(G))$. Then we have

$$\log L(s, \psi, E/\boldsymbol{Q})=\sum_{p} \sum_{m=1}^{\infty} \psi(p^m)/mp^{ms}=0 ,$$

where p runs over all prime numbers. If we put $b(n)$ as being $\psi(p^m)/m$ when $n=p^m$ and 0 otherwise, we obtain

$$\sum_{n=1}^{\infty} b(n)/n^s=0 \quad \text{for} \quad \mathrm{Re}\,(s)>1 .$$

Since the uniqueness theorem of Dirichlet series implies that $b(n)=0$ for all n, we have $\psi(p)=0$ for all p. By the Čebotarev's density theorem, every cyclic subgroup of G is a decomposition group for infinitely many prime numbers. Therefore, for every $\sigma \in G$, there exists an unramified $\mathfrak{P}$ such that $\sigma_{\mathfrak{P}}$ is conjugate to σ. Thus $\psi(\sigma)=0$ for all $\sigma \in G$, i.e.,

$$\psi=\sum_{\chi \in \mathrm{Irr}(G)} a(\chi)\chi=0 .$$

Hence $a(\chi)=0$ for all $\chi \in \mathrm{Irr}\,(G)$, since irreducible characters $\chi \in \mathrm{Irr}\,(G)$ are linearly independent. Q.E.D.

§3. Profinite groups

We shall consider some properties of a profinite group $\mathfrak{G}$. Let $\mathrm{Ch}(\mathfrak{G})$ be the $\boldsymbol{Z}$-module generated by characters of finite dimensional linear representations of $\mathfrak{G}$ over $\boldsymbol{C}$. An element of $\mathrm{Ch}(\mathfrak{G})$ is said to be a *generalized character*.

THEOREM 2. *For every profinite group* $\mathfrak{G}$,

$$\mathrm{Ch}(\mathfrak{G}) \cong \text{ind-lim}\, \mathrm{Ch}(\mathfrak{G}/\mathfrak{N})$$

as commutative ring, where $\mathfrak{N}$ *runs over all open normal subgroups of* $\mathfrak{G}$.

Proof. Suppose that ρ is a linear representation of $\mathfrak{G}$ of dimension n over $\boldsymbol{C}$; i.e., a continuous homomorphism of $\mathfrak{G}$ into $GL(n, \boldsymbol{C})$. Now there exists an open neighborhood U of the unity 1 in $GL(n, \boldsymbol{C})$ which does not include any subgroups except the identity subgroup $\{1\}$. By the continuity of ρ, the inverse image $\rho^{-1}(U)$ is open. Since $\mathfrak{G}$ is profinite, there exists an open subgroup $\mathfrak{H}$ included in $\rho^{-1}(U)$. Hence $\rho(\mathfrak{H}) \subset U$. The image $\rho(\mathfrak{H})$ is a subgroup of $GL(n, \boldsymbol{C})$ since ρ is a homomorphism. Thus it holds that $\rho(\mathfrak{H}) = \{1\}$. Therefore $\mathrm{Ker}\,\rho$ is open, as it includes an open subgroup $\mathfrak{H}$. Every linear representation ρ of $\mathfrak{G}$ can be regarded as that of the finite group $\mathfrak{G}/\mathrm{Ker}\,\rho$ in the standard way. Thus we easily see that for every $\chi \in \mathrm{Ch}(\mathfrak{G})$ there exists some open normal subgroup $\mathfrak{N}$ such that we can regard $\chi \in \mathrm{Ch}(\mathfrak{G}/\mathfrak{N})$. Conversely, the additive group $\mathrm{Ch}(\mathfrak{G}/\mathfrak{N})$ for every open normal subgroup $\mathfrak{N}$ can be embedded in $\mathrm{Ch}(\mathfrak{G})$ in the standard way and the family $(\mathrm{Ch}(\mathfrak{G}/\mathfrak{N}))$ for all open normal subgroups $\mathfrak{N}$ forms an inductive system. Therefore $\mathrm{Ch}(\mathfrak{G}) \cong \text{ind-lim}\,\mathrm{Ch}(\mathfrak{G}/\mathfrak{N})$ as additive group. The character theory of compact (finite) groups shows that every $\mathrm{Ch}(\mathfrak{G}/\mathfrak{N})$ and $\mathrm{Ch}(\mathfrak{G})$ are also commutative rings. Hence we easily check that the above isomorphism as additive group turns out to be an isomorphism as commutative ring. Q.E.D.

Let $\mathrm{Ch}(G)$ be the set of all generalized characters of a finite group G. For a subgroup H of G, let $\mathrm{Ind}_H^G\colon \mathrm{Ch}(H) \to \mathrm{Ch}(G)$ be the *induced mapping*, i.e., it is defined by

$$\mathrm{Ind}_H^G\, \psi(\sigma) = |H|^{-1} \sum \psi(\tau^{-1}\sigma\tau) \quad (\tau^{-1}\sigma\tau \in H,\ \tau \in G)$$

for $\psi \in \mathrm{Ch}(H)$ and every $\sigma \in G$. For a normal subgroup N, every generalized character of the quotient group G/N can be regarded as that of G in an obvious way. Here we easily check that the diagram

$$\begin{tikzcd}
\mathrm{Ch}(H/N) \arrow[r, "\mathrm{Ind}_{H/N}^{G/H}"] \arrow[d] & \mathrm{Ch}(G/N) \arrow[d] \\
\mathrm{Ch}(H) \arrow[r, "\mathrm{Ind}_H^G"] & \mathrm{Ch}(G)
\end{tikzcd}$$

is commutative, whenever N is a normal subgroup of G included in H.

We return to a profinite group $\mathfrak{G}$ again. Let $\mathfrak{H}$ be an open subgroup of $\mathfrak{G}$. The group $\mathfrak{H}$ is also a profinite group. For $\mathfrak{H}$ is a closed subgroup since $\mathfrak{H}$ is an open subgroup. Whenever $\mathfrak{N}$ is an open normal subgroup of $\mathfrak{G}$ included in $\mathfrak{H}$, the quotient groups $\mathfrak{G}/\mathfrak{N}$ and $\mathfrak{H}/\mathfrak{N}$ are finite groups, so the induced mapping $\mathrm{Ind}_{\mathfrak{H}/\mathfrak{N}}^{\mathfrak{G}/\mathfrak{N}}: \mathfrak{H}/\mathfrak{N} \to \mathfrak{G}/\mathfrak{N}$ has been defined. Then $(\mathrm{Ind}_{\mathfrak{H}/\mathfrak{N}}^{\mathfrak{G}/\mathfrak{N}})$ is a morphism of inductive systems, as we see the above. Thus from Theorem 2 there exists a mapping $\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}$ such that the diagram

$$\begin{array}{ccc} \mathrm{Ch}(\mathfrak{H}/\mathfrak{N}) & \xrightarrow{\mathrm{Ind}_{\mathfrak{H}/\mathfrak{N}}^{\mathfrak{G}/\mathfrak{N}}} & \mathrm{Ch}(\mathfrak{G}/\mathfrak{N}) \\ \downarrow & & \downarrow \\ \mathrm{Ch}(\mathfrak{H}) & \xrightarrow{\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}} & \mathrm{Ch}(\mathfrak{G}) \end{array}$$

is commutative.

§4. The absolute Galois group of Q and Artin L-functions

Let $\bar{Q}$ be an algebraic closure of the rational number field Q. Then every algebraic number field K can be considered as a subfield of $\bar{Q}$. The Galois group $\mathrm{Gal}(\bar{Q}/K)$, denoted by $\mathfrak{H}(K)$, is said to be the *absolute Galois group* of K. In particular, put $\mathfrak{G}=\mathfrak{H}(Q)$ from now on. Then every $\mathfrak{H}(K)$ can be considered as a subgroup of $\mathfrak{G}$. Since the group $\mathfrak{G}$ is a profinite group under the Krull topology, we can apply the consideration in Section 3 to the absolute Galois group $\mathfrak{G}$.

THEOREM 3. *Let $\mathfrak{L}$ be the set of Artin L-functions attached to all pairs, finite normal extensions of algebraic number fields and generalized characters of their Galois groups. Then;*

(i) *The set $\mathfrak{L}$ is a multiplicative group.*

(ii) *The additive group* $\mathrm{Ch}(\mathfrak{G})$ *is isomorphic to $\mathfrak{L}$ as abelian group under the mapping* $\Phi: \mathrm{Ch}(\mathfrak{G}) \to \mathfrak{L}$ *defined by*

$$\Phi(\psi)=L(s, \psi, E/Q)$$

for every $\psi \in \mathrm{Ch}(\mathfrak{G})$, where E is a subfield of $\bar{Q}$ corresponding to an open normal subgroup $\mathfrak{N}$ such that $\psi \in \mathrm{Ch}(\mathfrak{G}/\mathfrak{N})$.

Proof. From Theorem 2, there is such a subgroup $\mathfrak{N}$ as in (ii). The subfield E is a finite extension field over Q, as $\mathfrak{N}$ is open. Let E' be a subfield of $\bar{Q}$ corresponding to another open normal subgroup $\mathfrak{N}'$ such that $\psi \in \mathrm{Ch}(\mathfrak{G}/\mathfrak{N}')$. Then $\psi \in \mathrm{Ch}(\mathfrak{G}/\mathfrak{N} \cap \mathfrak{N}')$ and $\mathfrak{N} \cap \mathfrak{N}'$ is an open normal subgroup. Let E'' be a subfield of $\bar{Q}$ corresponding to $\mathfrak{N} \cap \mathfrak{N}'$. Then

$$L(s, \psi, E/Q)=L(s, \psi, E''/Q)=L(s, \psi, E'/Q) .$$

Thus the mapping Φ is well-defined. Let F/L be a finite normal extension of algebraic number fields and F' the Galois closure of $F/\boldsymbol{Q}$. Since

$$L(s, \varphi, F/L)=L(s, \varphi, F'/L)=L(s, \mathrm{Ind}_{\mathfrak{H}(L)}^{\mathfrak{G}} \varphi, F'/\boldsymbol{Q}) ,$$

every Artin L-function coincides with an Artin L-function attached to a finite normal extension over $\boldsymbol{Q}$. Thus we see at the same time that the set $\mathfrak{L}$ is a multiplicative group (Proof of (i)) and the mapping Φ is epimorphic as Artin L-function is additive. On the other hand, the mapping Φ is monomorphic from Theorem 2. The proof of (ii) is now complete. Q.E.D.

COROLLARY 1. *Let E/K and F/L be normal extensions of algebraic number fields. Let $\psi \in \mathrm{Ch}(\mathrm{Gal}(E/K))$ and $\varphi \in \mathrm{Ch}(\mathrm{Gal}(F/L))$. It holds that*

$$L(s, \psi, E/K)=L(s, \varphi, F/L)$$

if and only if

$$\mathrm{Ind}_{\mathfrak{H}(K)}^{\mathfrak{G}} \psi=\mathrm{Ind}_{\mathfrak{H}(L)}^{\mathfrak{G}} \varphi ,$$

where we regard as $\psi \in \mathrm{Ch}(\mathfrak{H}(K))$ and $\varphi \in \mathrm{Ch}(\mathfrak{H}(L))$ in the natural way.

Proof. The corollary directly follows from Theorem 3.

Two subgroups H, H' of a finite group G are said to be *Gassmann equivalent* when $|C(\sigma) \cap H|=|C(\sigma) \cap H'|$ for every conjugate class $C(\sigma)=\{\tau^{-1}\sigma\tau; \tau \in G\}$ in G. When ψ is an irreducible character of degree 1, i.e., a homomorphism of G into the multiplicative group of complex numbers, we call ψ a *linear character*. A character ψ of H is said to be *extendible* to G if there is a character of G whose restriction to H equals to ψ.

COROLLARY 2. *Suppose that ψ, φ are linear characters of* $\mathrm{Gal}(E/K)$, $\mathrm{Gal}(F/L)$ *respectively and both extensible to $\mathfrak{G}$. Then in order that*

$$L(s, \psi, E/K)=L(s, \varphi, F/L) ,$$

it is necessary and sufficient that:

(i) *Groups* $\mathrm{Gal}(M/K)$ *and* $\mathrm{Gal}(M/L)$ *are Gassmann equivalent in* $\mathrm{Gal}(M/\boldsymbol{Q})$, *where M is a finite normal extension field over $\boldsymbol{Q}$ including E and F;*

(ii) *Characters ψ and φ have the same extension.*

Proof. Put $G=\mathrm{Gal}(M/\boldsymbol{Q})$, $H(K)=\mathrm{Gal}(M/K)$, and $H(L)=\mathrm{Gal}(M/L)$. By Corollary 1, the equality

$$L(s, \psi, E/K)=L(s, \varphi, F/L)$$

is equivalent to

$$\mathrm{Ind}_{H(K)}^{G}\,\psi=\mathrm{Ind}_{H(L)}^{G}\,\varphi\ .$$

Then we easily obtain

$$|H(K)|^{-1}\psi(\sigma)\,|H(K)\cap C(\sigma)|=|H(L)|^{-1}\varphi(\sigma)\,|H(L)\cap C(\sigma)|$$

for every $\sigma\in G$. Putting $\sigma=1$, we get $|H(K)|=|H(L)|$. Thus we have for all $\sigma\in H(K)\cup H(L)$,

$$\psi(\sigma)=\varphi(\sigma)$$

and for all $\sigma\in G$,

$$|H(K)\cap C(\sigma)|=|H(L)\cap C(\sigma)|\ ,$$

because $\psi(\sigma)$ and $\varphi(\sigma)$ are roots of the unity. Hence an extension of ψ can be regarded as one of φ and $H(K)$, $H(L)$ are Gassmann equivalent in G. We can easily prove the converse, following along the opposite direction of the above. Q.E.D.

COROLLARY 3 (cf. Cassels & Fröhlich [3, p. 363 ex. 6.4.]). *The Zeta functions of algebraic number fields E and F are the same, i.e., $\zeta_E(s)=\zeta_F(s)$ if and only if $\mathrm{Gal}\,(M/E)$ and $\mathrm{Gal}\,(M/F)$ are Gassmann equivalent in $\mathrm{Gal}\,(M/\boldsymbol{Q})$, where M is a finite normal extension field over $\boldsymbol{Q}$ including E and F.*

Proof. Since

$$\zeta_E(s)=L(s,1,E/E)\quad\text{and}\quad\zeta_F(s)=L(s,1,F/F)\ ,$$

the corollary follows from Corollary 2.

§5. The finiteness of Artin L-functions

Let E/K be a finite normal extension of algebraic number fields and let $\psi\in\mathrm{Ch}\,(\mathrm{Gal}\,(E/K))$. We can write ψ as a form $\sum a(\chi)\chi$ for some $a(\chi)\in\boldsymbol{Z}$, where χ runs over all irreducible characters. Put

$$\mathrm{Ker}\,\psi=\bigcap_{a(\chi)\neq 0}\mathrm{Ker}\,\chi\ ,$$

where

$$\mathrm{Ker}\,\chi=\{\sigma\in\mathrm{Gal}\,(E/K);\ \chi(\sigma)=\chi(1)\}\ .$$

Then $\ker\psi$ is a normal subgroup. Let E_ψ be the fixed field of $\mathrm{Ker}\,\psi$. Then it follows from $\psi\in\mathrm{Ch}\,(\mathrm{Gal}\,(E_\psi/K))$ that

$$L(s,\psi,E/K)=L(s,\psi,E_\psi/K)\ .$$

THEOREM 4. *If a normal extension F/L and $\varphi\in\mathrm{Ch}\,(\mathrm{Gal}\,(F/L))$ are fixed, the set*

$$\{(\psi,E_\psi/K);\ L(s,\psi,E/K)=L(s,\varphi,F/L)\ \text{and } \psi \text{ is a character of a representation.}\}$$

is a finite set.

Remark 1. The set

$$\{(\psi, E/K);\ L(s, \psi, E/K)=L(s, \varphi, F/L)\ \text{and } \psi \text{ is a character of a faithful representation.}\}$$

coincides with the set in Theorem 4. For, if ρ is a representation of a group G with character ψ then $\operatorname{Ker}\rho=\operatorname{Ker}\psi$, so that ρ acts as a faithful representation of $G/\operatorname{Ker}\psi$ via G with character ψ.

Remark 2. The set

$$\{(\psi, E/K);\ L(s, \psi, E/K)=L(s, \varphi, F/L)\ \text{and } \psi \text{ is a character of a representation.}\}$$

is an infinite set. For, if E is a finite normal extension field over L including F then $(\varphi, E/L)$ is also contained in the above set.

Remark 3. The set

$$\{\psi\in \mathrm{Ch}\,(\mathrm{Gal}\,(F/L));\ L(s, \psi, F/L)=L(s, \varphi, F/L)\}$$

is not always a finite set. For example, comparing Table 1 with Table 2 in Section 6, we get for every $m\in \boldsymbol{Z}$,

$$L(s, \chi_1+m\chi_2-m\chi_3)=L(s, \chi_1)\ .$$

Proof of Theorem 4. Let M be the Galois closure of $F/\boldsymbol{Q}$. Let M' be the Galois closure of $ME_\psi/\boldsymbol{Q}$. The equalities

$$\mathrm{Ind}_{\mathfrak{H}(L)}^{\mathfrak{G}}\varphi(\sigma)=\mathrm{Ind}_{\mathfrak{H}(L)}^{\mathfrak{G}}\varphi(1)$$

hold for $\sigma\in\mathfrak{H}(M)$. On the other hand, it follows from Corollary 1 of Theorem 2 that

$$\mathrm{Ind}_{\mathfrak{H}(K)}^{\mathfrak{G}}\psi=\mathrm{Ind}_{\mathfrak{H}(L)}^{\mathfrak{G}}\varphi\ ,$$

and so, for all $\sigma\in\mathfrak{H}(M)$,

$$\mathrm{Ind}_{\mathfrak{H}(K)}^{\mathfrak{G}}\psi(\sigma)=\mathrm{Ind}_{\mathfrak{H}(K)}^{\mathfrak{G}}\psi(1)\ ,$$

i.e.,

$$\sum_{\substack{\tau^{-1}\sigma\tau\in\mathfrak{H}(K)/\mathfrak{H}(M')\\ \tau\in\mathfrak{G}/\mathfrak{H}(M')}}\psi(\tau^{-1}\sigma\tau)=\sum_{\substack{\tau^{-1}\sigma\tau\in\mathfrak{H}(K)/\mathfrak{H}(M')\\ \tau\in\mathfrak{G}/\mathfrak{H}(M')}}\psi(1)\ .$$

As ψ is a character of a representation, for all $\eta\in\mathfrak{H}(K)$,

$$|\psi(\eta)|\leqq\psi(1)\ .$$

Thus we easily have for all $\tau\in\mathfrak{H}$

$$\tau^{-1}\sigma\tau\in\mathfrak{H}(K)\ ,\quad \psi(\tau^{-1}\sigma\tau)=\psi(1)\ .$$

So

$$\mathfrak{H}(M) \subset \mathrm{Ker}\, \psi \subset \mathfrak{H}(K) .$$

Hence $\mathfrak{H}(M)=\mathfrak{H}(M')$, which implies $M=M'$. Therefore ψ can be regarded as a character of a representation the finite group $\mathrm{Gal}(M/\boldsymbol{Q})=\mathfrak{G}/\mathfrak{H}(M)$. Furthermore, the inequality

$$\psi(1) \leqq [K: \boldsymbol{Q}]\psi(1)=[L: \boldsymbol{Q}]\varphi(1)$$

implies that the degree of ψ is bounded. Since the number of representations of a finite group whose degree are bounded is finite, the theorem is proved. Q.E.D.

COROLLARY. *The number of algebraic number fields with the same Zeta function is finite.*

Proof. It is obvious from Theorem 4.

§6. Galois groups and Hecke L-functions

Let E/K and F/L be normal extensions of algebraic number fields with $G=\mathrm{Gal}(E/K)$ and $H=\mathrm{Gal}(F/L)$ respectively. Suppose now that there exists a one to one mapping Ψ of $\mathrm{Irr}(G)$ onto $\mathrm{Irr}(H)$ such that

$$L(s, \chi, E/K)=L(s, \Psi(\chi), F/L)$$

for all $\chi \in \mathrm{Irr}(G)$. Namely it follows from Corollary 1 of Theorem 3 that

$$\mathrm{Ind}_{\mathfrak{H}(K)}^{\mathfrak{G}} \chi=\mathrm{Ind}_{\mathfrak{H}(L)}^{\mathfrak{G}} \Psi(\chi)$$

for all $\chi \in \mathrm{Irr}(G)$. First we have to get

$$\Psi(1_G)=1_H .$$

For, since the equality

$$\mathrm{Ind}_{\mathfrak{H}(K)}^{\mathfrak{G}} 1_G=\mathrm{Ind}_{\mathfrak{H}(L)}^{\mathfrak{G}} \Psi(1_G)$$

holds, by the Frobenius reciprocity,

$$\begin{aligned}(\Psi(1_G), 1_H)_H &= (\mathrm{Ind}_{\mathfrak{H}(L)}^{\mathfrak{G}} \Psi(1_G), 1_{\mathfrak{G}})_{\mathfrak{G}}=(\mathrm{Ind}_{\mathfrak{H}(K)}^{\mathfrak{G}} 1_G, 1_{\mathfrak{G}})_{\mathfrak{G}} \\ &= (1_G, 1_G)_G=1 ,\end{aligned}$$

where $(\ ,\ )_{\mathfrak{G}}$ (resp. $(\ ,\ )_G$, $(\ ,\ )_H$) means the inner product with respect to the regular Haar measure on the compact group $\mathfrak{G}$ (resp. finite groups G, H). Thus $\Psi(1_G)=1_H$ as $\Psi(1_G)$ is irreducible. Next, the equality $\mathrm{Ind}_{\mathfrak{H}(L)}^{\mathfrak{G}} \Psi(\chi)=\mathrm{Ind}_{\mathfrak{H}(K)}^{\mathfrak{G}} \chi$ gives

$$[\mathfrak{G}: \mathfrak{H}(L)]\Psi(\chi)(1)=[\mathfrak{G}: \mathfrak{H}(K)]\chi(1) .$$

In particular, putting $\chi=1_G$, we get

$$[\mathfrak{G}: \mathfrak{H}(L)]=[\mathfrak{G}: \mathfrak{H}(K)] .$$

Therefore

$$\Psi(\chi)(1)=\chi(1)$$

for all $\chi\in\mathrm{Irr}\,(G)$. Now,

$$\begin{aligned}\mathrm{Ind}_{\mathfrak{H}(F)}^{\mathfrak{G}}1_{\mathfrak{H}(F)}&=\mathrm{Ind}_{\mathfrak{H}(L)}^{\mathfrak{G}}\,\mathrm{Ind}_{\mathfrak{H}(F)}^{\mathfrak{H}(L)}1_{\mathfrak{H}(F)}=\mathrm{Ind}_{\mathfrak{H}(L)}^{\mathfrak{G}}\,[\sum_{\theta\in\mathrm{Irr}(H)}\theta(1)\theta]\\&=\mathrm{Ind}_{\mathfrak{H}(L)}^{\mathfrak{G}}\,[\sum_{\chi\in\mathrm{Irr}(G)}\Psi(\chi)(1)\Psi(\chi)]\\&=\mathrm{Ind}_{\mathfrak{H}(K)}^{\mathfrak{G}}\,[\sum_{\chi\in\mathrm{Irr}(G)}\chi(1)\chi]=\mathrm{Ind}_{\mathfrak{H}(K)}^{\mathfrak{G}}\,\mathrm{Ind}_{\mathfrak{H}(E)}^{\mathfrak{H}(K)}1_{\mathfrak{H}(E)}\\&=\mathrm{Ind}_{\mathfrak{H}(E)}^{\mathfrak{G}}1_{\mathfrak{H}(E)}\,.\end{aligned}$$

Also

$$[\mathfrak{G}\colon\mathfrak{H}(E)]=[\mathfrak{G}\colon\mathfrak{H}(F)]\,,$$

so that

$$|G|=|H|\,.$$

It follows from

$$\mathrm{Ind}_{\mathfrak{H}(E)}^{\mathfrak{G}}1_{\mathfrak{H}(E)}=\mathrm{Ind}_{\mathfrak{H}(F)}^{\mathfrak{G}}1_{\mathfrak{H}(F)}\,,\quad \mathrm{Ind}_{\mathfrak{H}(K)}^{\mathfrak{G}}1_{\mathfrak{H}(K)}=\mathrm{Ind}_{\mathfrak{H}(L)}^{\mathfrak{G}}1_{\mathfrak{H}(L)}$$

that

$$\zeta_E(s)=\zeta_F(s)\,,\quad \zeta_K(s)=\zeta_L(s)$$

and further that if E and K are normal extension fields over $\boldsymbol{Q}$, then $E=F$ and $K=L$ (of course $G=H$) since $\mathfrak{H}(E)$ and $\mathfrak{H}(K)$ are normal subgroups of $\mathfrak{G}$. Now since the number of linear characters of G equals to that of H, it holds that

$$|G/G'|=|H/H'|\,,$$

where G' and H' are the commutator groups of G and H respectively. If G is an abelian group, then so is H and $|G|=|H|$. But we can not obtain $G\cong H$ even in this case, as the following example shows.

LEMMA 1. *Let G_1 be a group of order 16 generated by elements P, Q, R with relations;*

$$P^4=1\,,\quad Q^4=1\,,\quad R^2=1\,,\quad Q^{-1}PQ=P^{-1}$$
$$P^2=Q^2\,,\quad R^{-1}QR=Q\,,\quad R^{-1}PR=P\,.$$

Then the group G_1 has exactly ten conjugate classes and the following character table. (It is Table 1 in the next page.)

Proof. The commutator group of G_1 is the cyclic subgroup $\langle P^2\rangle$ of order 2. For the group G_1 is not commutative and $G_1/\langle P^2\rangle$ is an elementary abelian 2-group of order 8. Thus the group G_1 has exactly eight linear characters. Since $|G_1|=\sum\chi(1)^2$ $(\chi\in\mathrm{Irr}\,(G_1))$, the group G_1 has two irreducible characters of degree 2 and exactly ten conjugate classes. We easily obtain $\chi_0,\cdots,\chi_7$ from characters of the abelian

Table 1

	1	P^2	R	P^2R	$P^{\pm 1}$	$P^{\pm 1}Q$	$P^{\pm 1}R$	$P^{\pm 1}QR$	Q, P^2Q	QR, P^2QR
exp.	1	2	2	2	4	4	4	4	4	4
Card.	1	1	1	1	2	2	2	2	2	2
χ_0	1	1	1	1	1	1	1	1	1	1
χ_1	1	1	1	1	-1	-1	-1	-1	1	1
χ_2	1	1	1	1	1	-1	1	-1	-1	-1
χ_3	1	1	1	1	-1	1	-1	1	-1	-1
χ_4	1	1	-1	-1	1	1	-1	-1	1	-1
χ_5	1	1	-1	-1	1	-1	-1	1	-1	1
χ_6	1	1	-1	-1	-1	-1	1	1	1	-1
χ_7	1	1	-1	-1	-1	1	1	-1	-1	1
χ_8	2	-2	2	-2	0	0	0	0	0	0
χ_9	2	-2	-2	2	0	0	0	0	0	0

group $G_1/\langle P^2\rangle$. Suppose that χ is an irreducible character of degree 2. Put

$$Z(\chi)=\{\sigma\in G_1;\ |\chi(\sigma)|=\chi(1)\}\ .$$

Then $Z(\chi)$ coincides with the center $Z(G_1)=\langle P^2, R\rangle$ of G_1. For in general, it holds that $Z(\chi)\supset Z(G_1)$. Conversely, as χ is irreducible, $(\chi, \chi)_{G_1}=1$, in other words,

$$\sum|\chi(\sigma)|^2=16 \quad (\sigma\in G_1)\ .$$

Thus it follows that

$$|\chi(\sigma)|=2 \quad \text{for all} \quad \sigma\in Z(G_1)$$

and

$$\chi(\sigma)=0 \quad \text{for all} \quad \sigma\in G_1\backslash Z(G_1)\ .$$

Hence $Z(\chi)=Z(G_1)$. Next, from $(\chi, \chi_0)_{G_1}=0$,

$$\chi(P^2)+\chi(R)+\chi(P^2R)=-2$$

and from $(\chi, \chi_4)_{G_1}=0$,

$$\chi(P^2)-\chi(R)-\chi(P^2R)=-2\ .$$

Therefore,
$$\chi(P^2)=-2\ ,$$
$$\chi(R)+\chi(P^2R)=0\ ,$$

and

$$|\chi(R)|=|\chi(P^2R)|=2\ .$$

On the other hand,

$$\overline{\chi(R)}=\chi(R^{-1})=\chi(R)$$

and

$$\overline{\chi(P^2R)}=\chi((P^2R)^{-1})=\chi(P^2R)\ ,$$

where — means the complex conjugation. Hence we have

$$\chi_8(P^2)=-2\ ,\quad \chi_8(R)=2\ ,\quad \chi_8(P^2R)=-2$$

and

$$\chi_9(P^2)=-2\ ,\quad \chi_9(R)=-2\ ,\quad \chi_9(P^2R)=2\ . \qquad \text{Q.E.D.}$$

LEMMA 2. *Let G_2 be a group of order* 16 *generated by elements S, T with relations;*

$$S^4=1\ ,\quad T^4=1\ ,\quad T^{-1}ST=S^3\ .$$

Then the group G_2 has exactly ten conjugate classes and the following character table;

Table 2

	1	S^2	T^2	S^2T^2	$S^{\pm1}$	$S^{\pm1}T$	$S^{\pm1}T^2$	$S^{\pm1}T^3$	T, S^2T	T^3, S^2T^3
exp.	1	2	2	2	4	4	4	4	4	4
card.	1	1	1	1	2	2	2	2	2	2
θ_0	1	1	1	1	1	1	1	1	1	1
θ_1	1	1	1	1	-1	-1	-1	-1	1	1
θ_2	1	1	1	1	1	-1	1	-1	-1	-1
θ_3	1	1	1	1	-1	1	-1	1	-1	-1
θ_4	1	1	-1	-1	1	i	-1	$-i$	i	$-i$
θ_5	1	1	-1	-1	1	$-i$	-1	i	$-i$	i
θ_6	1	1	-1	-1	-1	$-i$	1	i	i	$-i$
θ_7	1	1	-1	-1	-1	i	1	$-i$	$-i$	i
θ_8	2	-2	2	-2	0	0	0	0	0	0
θ_9	2	-2	-2	2	0	0	0	0	0	0

$(i=\sqrt{-1})$

Proof. For the proof we employ the same technique as in Lemma 1.

The Groups G_1 in Lemma 1 and G_2 in Lemma 2 have the same order 16, so each is embedded in the symmetric group S_{16} of degree 16 via its regular representation. Here, if two elements of G_1 and G_2 have the same exponent, then they have the same cyclic structure.

In general, if elements of the symmetric group have the same cyclic structure, then they are conjugate to each other. Therefore we can easily calculate characters of S_{16} induced from irreducible characters of the subgroups G_1 and G_2. The results are in the following table;

Table 3

Exponent	1	2	4	
Cyclic structure	(1)	$\prod_{i=1}^{8}(a_{i1}a_{i2})$	$\prod_{i=1}^{4}(a_{i1}a_{i2}a_{i3}a_{i4})$	otherwise
$\chi_0^*=\theta_0^*$	15!	$3\times 2^7\times 7!$	$18\times 4!^4$	0
$\chi_1^*=\chi_2^*=\chi_3^*=\theta_1^*=\theta_2^*=\theta_3^*$	15!	$3\times 2^7\times 7!$	$-6\times 4!^4$	0
$\chi_4^*=\chi_5^*=\chi_6^*=\chi_7^*=\theta_4^*=\theta_5^*=\theta_6^*=\theta_7^*$	15!	$-2^7\times 7!$	0	0
$\chi_8^*=\chi_9^*=\theta_8^*=\theta_9^*$	$2\times 15!$	$-2^8\times 7!$	0	0

(* means an induced character.)

The group S_{16} may be realized as a Galois group Gal(E/K) of a normal extension E/K of algebraic number fields. Let K_1 and K_2 be the fixed fields of the subgroups G_1 and G_2 respectively. From the above tables, as to Artin L-functions on relative normal extensions E/K_1 and E/K_2, it follows that

$$L(s, \chi_i, E/K_1)=L(s, \theta_i, E/K_2)$$

for $i=0, 1, \cdots, 9$. But two Galois groups Gal(E/K_1), Gal(E/K_2) are not isomorphic, because Gal$(E/K_1)=G_1$ and Gal$(E/K_2)=G_2$ from the definitions.

Now, the quotient group $G_1/\langle P\rangle$ is an elementary abelian 2-group of order 4 and the quotient group $G_2/\langle S\rangle$ is an cyclic group of order 4. Then irreducible characters of $G_1/\langle P\rangle$ are exhausted by χ_0, χ_2, χ_4, χ_5 and also those of $G_2/\langle S\rangle$ by θ_0, θ_2, θ_4, θ_5. Let E_1 and E_2 be the fixed fields of the subgroups $\langle P\rangle$ and $\langle S\rangle$ respectively. Therefore it follows from the above tables that

$$L(s, \chi_i, E_1/K_1)=L(s, \theta_i, E_2/K_2)$$

for $i=0, 2, 4, 5$. In case of an abelian extension, if ψ is a linear character of its Galois group and φ is the corresponding congruence class character, then the Artin L-function for ψ coincides with the so-called Hecke L-function for φ. Summing up our results we obtain the last theorem.

THEOREM 5. *There exist relative abelian extensions* E_1/K_1 *and*

E_2/K_2 of algebraic number fields such that each Hecke L-function on E_1/K_1 coincides with that on E_2/K_2 and yet two Galois groups of E_1/K_1 and E_2/K_2 are not isomorphic.

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